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MINIMAL WEIGHT DESIGN FOR A BUILT-IN BEAM

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PREFACE

In a previous Memorandum, RM—2887—PR, "Minimum—Weight
Design for Moving Loads" by O. Gross and W. Prager, a problem
involving a horizontal I—beam which is built in at one end and simply
supported at the other was proposed and solved.   The rigorous
mathematical treatment is contained in RM—2993—PR, "A Linear
Program of Prager's: Notes on Linear Programming and Ex—
tensions — Part 60." The present Memorandum considers a second
beam—design problem, namely, the problem of a beam built in at
both ends and subject to the same loading condition, i.e., a con—
centrated vertical unit load moving slowly along the span.

Beam designs of the type (variable cross section) considered
in this Memorandum are perhaps difficult to manufacture.   They
nevertheless represent, in a sense, the ultimate in weight saving
and are therefore at least of theoretical importance, particularly
in the view of possible future application to interplanetary structures
in which excess weight could spiral transportation costs.

SUMMARY

This Memorandum treats the problem of the design of a horizontal I—beam of variable flange thickness and constant web height. The beam is built in at both ends and is otherwise unsupported, but is subjected to a concentrated vertical unit load that moves slowly from one end to the other. A linear program in function space is obtained, using the concepts of limit analysis (based on the signum—type stress—strain diagram), and the solution of the program is given, yielding the minimum weight design as well as the critical reactions at the ends of the beam.

CONTENTS

MINIMAL WEIGHT DESIGN FOR A BUILT-IN BEAM

1. FORMULATION OF THE PROBLEM

Some time ago W. Prager proposed the following problem.

An I-beam is to be designed to fulfill these specifications: The beam is to be built in at both ends to a solid structure. The web is to have constant height and thickness. A vertical concentrated load of fixed intensity is assumed to move slowly across the span. Assuming that the bending moment at each cross section is taken up by the flanges, which are allowed to vary in cross-sectional area, what distribution of flange cross section will yield a minimal weight design, subject to the constraint that the beam will support the load in plastic flow?

There is no loss of generality in assuming the beam to be of unit length. In order to phrase the problem in mathematical terms, it is helpful to have recourse to the following Fig. 1, depicting the idealized beam together with its variable load and (statically indeterminate) reactions. We assume, also, without loss in generality, that the moving load is of unit intensity.

Fig. I
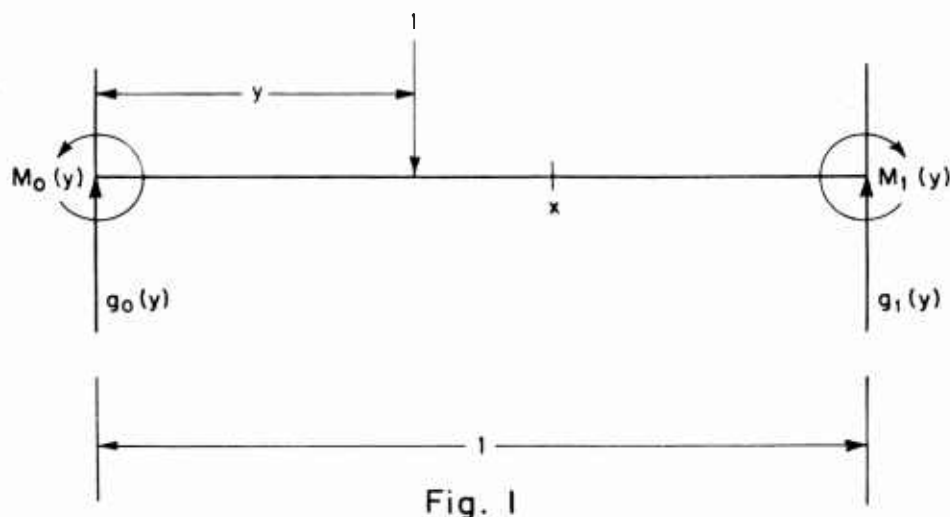
Since the load is moving sufficiently slowly to obviate dynami—
cal considerations, the beam will maintain static equilibrium.
Imposing the force and moment conditions in terms of the load and
the indeterminate reactions g_i, M_i at the ends (which, for a fixed
design, are functions of the position of the load) then gives

(1.1) $$g_o(y) + g_1(y) = 1,$$

(1.2) $$y - M_o(y) + M_1(y) - g_1(y) = 0.$$

(Moments are taken counterclockwise at $x = 0$.)

Next, we determine the criticality conditions. If the critical
design bending moment of the beam at a variable position x is
denoted by $f(x)$, then in order that the beam will sustain the load
in an arbitrary position, we require that the critical bending
moment shall nowhere be exceeded in absolute value. Two cases
present themselves.

Case 1. $0 \leq y \leq x \leq 1$ (as depicted in Fig. 1). In this case,
the bending moment at x is given (looking at the section of the
beam to the left of x and taking moments counterclockwise) by

$$M(x, y) = x - y - xg_o(y) + M_o(y).$$

Thus, for this case, the criticality condition becomes

(1.3) $$f(x) \geq |x - y - xg_o(y) + M_o(y)| \text{ if } 0 \leq y \leq x \leq 1,$$

where the bars denote absolute value.

Case 2. $0 \leq x \leq y \leq 1$. In this case, upon making the appropriate modification in Fig. 1, for the bending moment at x (looking at the section of the beam to the right of x and taking moments clockwise) we obtain

$$M(x, y) = y - x - (1 - x) g_1(y) + M_1(y).$$

From the equilibrium conditions (1.1) and (1.2) we can express M in terms of the reactions at 0; thus, a little algebra gives

$$M(x, y) = -x g_0(y) + M_0(y)$$

(which we could have derived anyway, of course, on sighting to the left of x). Thus, for this case, the criticality condition becomes

$$(1.4) \qquad f(x) \geq \left| -x g_0(y) + M_0(y) \right| \text{ if } 0 \leq x \leq y \leq 1.$$

Next, observe that (1.3) and (1.4) can be combined into an equivalent two—parameter family of constraint conditions, namely

$$(1.5) \qquad f(x) \geq \left| \max(0, x - y) - x g(y) + M(y) \right| \ 0 \leq \frac{x}{y} \leq 1.$$

(We have dropped the subscript 0 for convenience.)

Finally, since the web of the idealized beam is of constant height, the design bending moment at x will be proportional to the cross—sectional area of the flanges there. Hence, the variable portion of the <u>weight</u> of the beam will be proportional to

$\displaystyle\int_0^1 f(x)\,dx.$ [*]   Since, moreover, it follows from a fundamental principle of limit analysis that if there exist reactions M, g such that the bending moment is nowhere exceeded, then the beam will sustain the load in plastic flow. Thus, the mathematical formulation of the design problem can be phrased as follows.

Find real measurable functions f, g, M on [0, 1] such that

$$\int_0^1 f(x)\,dx \;=\; \text{minimum},$$

subject to

(1.6) $\qquad f(x) \geq \big|\,\max(0,\; x - y) - xg(y) + M(y)\big|,\quad 0 \leq \dfrac{x}{y} \leq 1.$

The solution to this (continuous linear programming) problem will be described in the following section. Feasibility and minimality will be verified in Secs. 3 and 4, respectively.

2. DESCRIPTION OF THE SOLUTION

We now assert that the following is a solution to the design problem as formulated at the end of Sec. 1:

(2.1) $\quad f(x) = \dfrac{x}{2} + \dfrac{1 - 2x}{2}\;\log\sqrt{2}(1 - 2x)\quad\text{if }\;0 \leq x \leq \dfrac{1}{4},$

$$= \frac{\sqrt{(1 - 2x)^2 + 2} - 1}{4} - \frac{1}{2}(2x - 1)\sinh^{-1}\left(\frac{2x - 1}{\sqrt{2}}\right)$$

$$\text{if }\;\frac{1}{4} < x \leq \frac{3}{4},$$

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[*] Assuming the material to be of uniform strength and density.

and otherwise

$$f(x) = f(1-x) \quad ;$$

(2.2) $\qquad g(y) = \frac{1}{2} - \sinh^{-1}\left(\frac{2y-1}{\sqrt{2}}\right) \quad \text{if } \frac{1}{4} \leq y \leq \frac{3}{4} \quad ;$$

and

(2.3) $\qquad M(y) = \frac{y}{2} - \frac{1}{2}\sinh^{-1}\left(\frac{2y-1}{\sqrt{2}}\right) - \frac{\sqrt{(1-2y)^2 + 2} - 1}{4}$

$$\text{if } \frac{1}{4} \leq y \leq \frac{3}{4} \quad .$$

The value of the program is given by

(2.4) $\qquad \displaystyle\int_{0}^{1} f(x)dx = \frac{1}{8} \log 2 \approx 0.08664.$

Graphs of these functions are appended at the end of this paper. A few remarks are in order regarding the proposed solution and its graphs.

Remark 1.

We observe from the formulas and graphs (particularly Fig. 2) that the solution is symmetric. This is not surprising, since it can be verified a priori from the symmetry of the model and linearity that if there exists a solution at all, there exists a symmetric one. Thus, in seeking a solution, we could have imposed with impunity the condition

$$M_o(y) = M_1(1-y)$$

(upon reverting to an original notation of Fig. 1). Using the external equilibrium conditions (1.1) and (1.2), we obtain

$$g_0(y) = 1 - y + M_0(y) - M_0(1 - y).$$

Dropping the subscript 0, as before, and substituting in the constraint condition (1.6), we have

$$f(x) \geq \left| \max(0, \ x - y) - x + xy + xM(1 - y) + (1 - x) M(y) \right|,$$

which involves only one unknown reaction function. Thus, if $x \geq y$, we must have

$$f(x) \geq \left| -(1 - x)y + xM(1 - y) + (1 - x) M(y) \right|,$$

and if $x \leq y$,

$$f(x) \geq \left| -(1 - y)x + xM(1 - y) + (1 - x) M(y) \right|.$$

These last two inequalities are, in effect, in the form of the constraint conditions in the mathematical formulation as obtained by Prager.

Remark 2.

Notice that we have not defined the g and M functions outside the interval $[1/4, \ 3/4]$ (Figs. 3 and 4). The reason for this is that the beam does not attain critical condition until the load reaches the middle section of the beam. In other words, the reactions are relatively indeterminate up to this point; i.e., infinitely many feasible extensions of these functions are possible outside the critical range. We shall not characterize all of them as was done in [2] , since we need only exhibit one such extension, as is required for the verification of feasibility in the subsequent section. As it turns out, simply extending the functions analytically is not feasible.

Remark 3.

The dotted lines in Fig. 2 constitute the graph of the bending moment in the beam (as reflected across the x—axis) corresponding to a typical position of the load. The beam is critical at <u>three</u> points (indicated by the dots): two points of tangency in the outer sections, and one at the point of application of the load. The graph of these critical points is illustrated in Fig. 5. The two curved arcs shown are from the two branches of the hyperbola $y = x - \dfrac{1}{4(2x-1)}$.

3. VERIFICATION OF FEASIBILITY

In this section we shall give a partial verification that with an appropriate extension of the given g and M functions, the con—straints (1.6) are met by the solution in the preceding section. To this end, we extend M outside $[1/4, \ 3/4]$ as the continuous piece—wise linear extension such that $M(0) = M(1) = 0$. This extension is shown by the dotted lines in Fig. 4. Imposing the symmetry con—dition

$$g(y) = 1 - y + M(y) - M(1 - y)$$

gives rise to the linear extensions of g shown by the dotted lines in Fig. 3.

We are now required to verify that the inequality

$$f(x) \geq \big| \max(0, \ x - y) - xg(y) + M(y) \big|, \quad 0 \leq \frac{x}{y} \leq 1,$$

is satisfied by our proposed solution. Our first observation is that because of the symmetry of our solution, it is sufficient to verify this inequality for $y \in [0, \ 1/2]$. Thus, to check it outside the

interval $\left(1/4,\ 3/4 \right)$, we need only consider the case $0 \leq y \leq 1/4$.
For this case however, and for any given value of x, the function
of y under the absolute value sign is, by our extension, piecewise
linear in at most two pieces, and hence will attain its maximum
absolute value at one of the points $y = 0$, $y = x$ or $y = 1/4$. Since
$g(0) = 1$ and $M(0) = 0$, the case $y = 0$ reduces to $f(x) \geq 0$. We
leave the straightforward verification of this inequality as an
exercise. Next, if $y = x$, then in this case we must have
$0 \leq x \leq 1/4$, and hence we are required only to verify that

$$f(x) \geq \left| -xg(x) + M(x) \right| \quad \text{if} \quad 0 \leq x \leq 1/4.$$

But one checks for this case that the function on the right is simply
a quadratic function which vanishes at $x = 0$ and $x = 1$ and hence
will attain its maximum on the specified range at $x = 1/4/$ On the
other hand, since f is decreasing on this interval, in order to
check the inequality it suffices to check it at $x = 1/4$; i.e., it
suffices to verify that

$$f\left(\frac{1}{4} \right) \geq \left| -\frac{1}{4} g \left(\frac{1}{4} \right) + M\left(\frac{1}{4} \right) \right| .$$

However, using formulas (2.1), (2.2), and (2.3), which now apply,
readily gives

$$f\left(\frac{1}{4} \right) = \left| -\frac{1}{4} g\left(\frac{1}{4} \right) + M\left(\frac{1}{4} \right) \right| .$$

This disposes of the case $y = x \leq \frac{1}{4}$.

Finally, we note that the case $y = 1/4$ is included by continuity in the range $[1/4, \ 3/4]$. Thus, to complete the verification it is sufficient to establish that

$$f(x) \geq |\max(0, x - y) - xg(y) + M(y)| \quad \text{if} \ \ 0 \leq x \leq 1$$

and $1/4 \leq y \leq 3/4$. In this range, however, formulas (2. 2) and (2. 3) apply. Now observe, that for a given value of y in this range, the function of x under the absolute value sign is piecewise linear (in at most two pieces). We assert that f is simply the <u>absolute upper envelope</u> of this family of functions, i.e.,

$$f(x) = \max_{\frac{1}{4} \leq y \leq \frac{3}{4}} |\max(0, \ x - y) - xg(y) + M(y)| \quad 0 \leq x \leq 1.$$

Since the evaluation of the right member of the above was precisely the method by which the function f was obtained, we shall leave this elementary albeit arduous calculus exercise to the reader, lest this section become as long as the corresponding one in [2].

4. VERIFICATION OF MINIMALITY

In this final section we shall show th. .t the proposed feasible solution given by (2. 1)—(2. 3) satisfies the minimality requirement and hence solves the problem proposed in Sec. 1.

Let $J^* = \dfrac{1}{4} \displaystyle\int_{\frac{3}{4}}^{1} \dfrac{dx}{2x - 1}$. It is then a trivial matter to check that

$$J^* = \frac{1}{8} \log 2.$$

Next, by means of well known identities on the inverse hyperbolic sine, it becomes a straightforward exercise to verify that the functions f, g, M given in Sec. 2 satisfy the following relations:[*]

$$(4.1) \quad \begin{cases} f(x) = -xg\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) + M\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) & \text{if } 0 \le x \le \dfrac{1}{4}, \\[2em] = xg(x) - M(x) & \text{if } \dfrac{1}{4} \le x \le \dfrac{3}{4}, \\[2em] = \dfrac{\frac{1}{4}}{2x-1} - xg\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) + M\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) & \text{if } \dfrac{3}{4} \le x \le 1. \end{cases}$$

Consequently, we have

$$(4.2) \quad \int_0^1 f(x)dx = \int_0^{\frac{1}{4}} \left(-xg\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) + M\left(x - \dfrac{\frac{1}{4}}{2x-1}\right)\right) dx$$

$$+ \int_{\frac{1}{4}}^{\frac{3}{4}} (xg(x) - M(x))dx$$

$$+ J* + \int_{\frac{3}{4}}^1 \left(-xg\left(x - \dfrac{\frac{1}{4}}{2x-1}\right) + M\left(x - \dfrac{\frac{1}{4}}{2x-1}\right)\right) dx.$$

Upon making the changes of variable yielding $x' = x - 1/4\,(2x - 1)$ in the 1st and 3rd integrals in the right member of the above, we obtain, respectively (using the appropriate branches of the hyperbola)

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[*]Notice that the values of the arguments of the g and M functions used lie in the appropriate range $(1/4,\ 3/4)$.

$$\int_0^{\frac{1}{4}} \left(-\, xg\left(x - \frac{\frac{1}{4}}{2x-1} \right) + M\left(x - \frac{\frac{1}{4}}{2x-1} \right) \right) dx$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} \left\{ \left(-\frac{2x'+1}{4} + \frac{\sqrt{4x'^2 - 4x' + 3}}{4} \right)\, g(x') + M(x') \right\} \cdot$$

$$\left\{ \frac{1}{2} - \frac{x' - \frac{1}{2}}{\sqrt{4x'^2 - 4x' + 3}} \right\} \quad dx',$$

and

$$\int_{\frac{3}{4}}^{1} \left(-xg\left(x - \frac{\frac{1}{4}}{2x-1} \right) + M\left(x - \frac{\frac{1}{4}}{2x-1} \right) \right) dx$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} \left\{ \left(-\frac{2x'+1}{4} - \frac{\sqrt{4x'^2 - 4x' + 3}}{4} \right)\, g(x') + M(x') \right\} \cdot$$

$$\left\{ \frac{1}{2} + \frac{x' - \frac{1}{2}}{\sqrt{4x'^2 - 4x' + 3}} \right\} dx'.$$

Using the fact that the two outer integrals are now expressed as integrals over the same interval $\left(1/4,\ 3/4 \right)$, we can add their integrands, collecting coefficients of $g(x')$ and $M(x')$, to obtain as their joint contribution:

$$\int_{\frac{1}{4}}^{\frac{3}{4}} (-x'\, g(x') + M(x'))\, dx' \ .$$

Upon substituting this joint contribution in (4.2) we obtain

$$(4.3) \quad \int_0^1 f(x)dx = J^* + \int_{\frac{1}{4}}^{\frac{3}{4}} (xg(x) - M(x))dx + \int_{\frac{1}{4}}^{\frac{3}{4}} (-x'g(x') + M(x'))dx'$$

$$= J^*,$$

since the integrals involving the reactions cancel.

Next, let $(\tilde{f}, \tilde{g}, \tilde{M})$ denote any feasible triple of functions satisfying the constraints (1.6), i.e.,

$$(4.2) \qquad \tilde{f}(x) \geq |\max(0, x - y) - x\tilde{g}(y) + \tilde{M}(y)|, \quad 0 \leq \frac{x}{y} \leq 1.$$

In particular, we must have

$$(4.3) \qquad \tilde{f}(x) \geq \max(0, x - y) - x\tilde{g}(y) + \tilde{M}(y), \qquad 0 \leq \frac{x}{y} \leq 1.$$

Now, one verifies that on the interval $(0, 1/4)$

$$(4.4) \qquad 0 \leq x \leq x - \frac{\frac{1}{4}}{2x - 1} \leq 1.$$

Hence (4.3) must hold for $y = x - \dfrac{\frac{1}{4}}{2x - 1}$, i.e., using (4.4), we must have

$$(4.5) \qquad \tilde{f}(x) \geq - x\tilde{g}\left(x - \frac{\frac{1}{4}}{2x - 1}\right) + \tilde{M}\left(x - \frac{\frac{1}{4}}{2x - 1}\right) \quad \text{if } 0 \leq x \leq \frac{1}{4}.$$

Next, (4.2) implies

$$\tilde{f}(x) \geq - \max(0, \ x - y) + x\tilde{g}(y) - \tilde{M}(y), \quad 0 \leq \frac{x}{y} \leq 1$$

which must hold for $y = x$. In particular, on the interval $(1/4, \ 3/4)$, we must have

(4.6) $\quad \tilde{f}(x) \geq x\tilde{g}(x) - \tilde{M}(x) \ $ if $ \ \frac{1}{4} \leq x \leq \frac{3}{4}$.

Finally, one verifies that on the interval $(3/4, \ 1)$,

(4.7) $$0 \leq x - \frac{\frac{1}{4}}{2x - 1} \leq x \ .$$

Hence, on this interval (4.3) must hold with $y = x - 1/4(2x - 1)$; that is,

(4.8) $$\tilde{f}(x) \geq \frac{\frac{1}{4}}{2x - 1} - x\tilde{g}\left(x - \frac{\frac{1}{4}}{2x - 1}\right) + \tilde{M}\left(x - \frac{\frac{1}{4}}{2x - 1}\right)$$

$$\text{if } \frac{3}{4} \leq x \leq 1.$$

Now observe that (4.5), (4.6), and (4.8) are precisely the relations (4.1) with "=" replaced by "$\geq$". Consequently, of one integrates the inequalities over their ranges, adds them together, and makes the same changes of variable as before, the integrals involving the unspecified reactions $\tilde{g}', \tilde{M}$ will cancel to obtain

$$\int_0^1 f(x)dx \geq J^* = \int_0^1 f(x)dx.$$

Thus the design is indeed minimal and the verification is complete.

Perhaps a word or two is in order regarding the almost com—pletely synthetic treatment of beam—design problems, since the reader is no doubt wondering how the author came to obtain the obviously complicated solution in the first place. The author need not apologize for this insofar as he has indicated a complete veri—fication. Suffice it to say, however, that the "method" of derivation involved quite a bit of judicious guessing, solving of differential equations, matching end conditions, etc. Lest this Memorandum become unduly long, we shall not go into details here, but rather refer the reader to [1], where motivation for solving a similar beam—design problem is given.

We shall close this Memorandum with some remarks about the loads that our beam design will sustain. Let W denote an arbitrary downward static—load distribution, of total weight not exceeding unity, i.e., $dW(y) \geq 0$ and $\int_0^1 dW(y) \leq 1$. We assert that our beam will support this load. This is perhaps intuitively clear. Nevertheless, we shall present a rigorous verification. The appropriate modification of Fig. 1 yields the following as the criticality condition corresponding to (1.6):

$$(4.9) \qquad f(x) \geq \left| \int_0^x (x - y)\, dW(y) - x g_L(W) + M_L(W) \right|, \quad 0 \leq x \leq 1,$$

where $g_{L(W)}$ and $M_L(W)$ are the support and moment reactions, induced by the load, at the left end of the beam. Thus, the beam will sustain W if there exist nonnegative functionals g_L and M_L satisfying (4.9). Now, we have already verified that

$$f(x) \geq \max(0,\, x - y) - x g(y) + M(y), \quad 0 \leq \frac{x}{y} \leq 1.$$

Integrating this last (over $[0, 1]$) with respect to $dW(y) \geq 0$ and noting $f(x) \geq 0$ and $\int_0^1 dW(y) \leq 1$, we obtain

$$f(x) \geq \int_0^x (x - y) \, dW(y) - x \int_0^1 g(y) \, dW(y) + \int_0^1 M(y) \, dW(y).$$

Similarly, using the established inequality

$$f(x) \geq - \max(0, \ x - y) + xg(y) - M(y),$$

we obtain

$$f(x) \geq - \int_0^x (x - y) \, dW(y) + x \int_0^1 g(y) \, dW(y) - \int_0^1 M(y) \, dW(y).$$

Thus, it suffices to set

$$g_L(W) = \int_0^1 g(y) \, dW(y)$$

and

$$M_L(W) = \int_0^1 M(y) \, dW(y).$$

Combining the two inequalities then gives (4.9). Hence the beam will sustain the load. Moreover, since criticality is not reached, for example, by a unit concentrated load until it reaches its mid section, the beam will support other loads as well. We shall refrain, however, from giving a complete characterization. Thus, in effect, we have solved a more general problem than the one proposed! We can, moreover, phrase our result in another way. Let $\mathcal{F}$ be a family of downward load distributions over $[0, 1]$, i.e., $W \in \mathcal{F} \Rightarrow dW(y) \geq 0$ if $0 \leq y \leq 1$. Then there exists a

built—in beam design capable of supporting any $W \in \mathcal{F}$ such that its design bending moment ϕ satisfies

$$\int_0^1 \phi(x)\,dx \leq m \sup_{W \in \mathcal{F}} \int_0^1 dW(y),$$

where $m = \frac{1}{8} \log 2$, (provided the right member is finite), the constant m being best possible.

A curious research problem suggests itself at this point: Given a built—in beam design B of specified variable cross section, find a beam design of minimal weight which will sustain every down—ward static—load distribution that B does. It is not immediately clear, for example, that B is itself minimal. We note in passing, however, that for the case of a beam that is <u>simply supported</u> at each end and of <u>concave</u>[*] design moment, the assertion is true.

As an amusing sidelight on the foregoing remarks, consider the situation in which two men of weight one—half unit each (Martians, no doubt) are walking across our beam in opposite directions. Since the beam will not become critical unless they happen to pass each other in its midriff, the little fellows should be careful to be out of step in case of such an event.

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[*] More generally, if its (continuous) graph is "star shaped" with respect to the points $(0, 0)$ and $(1, 0)$, then this last con—dition is both necessary and sufficient.
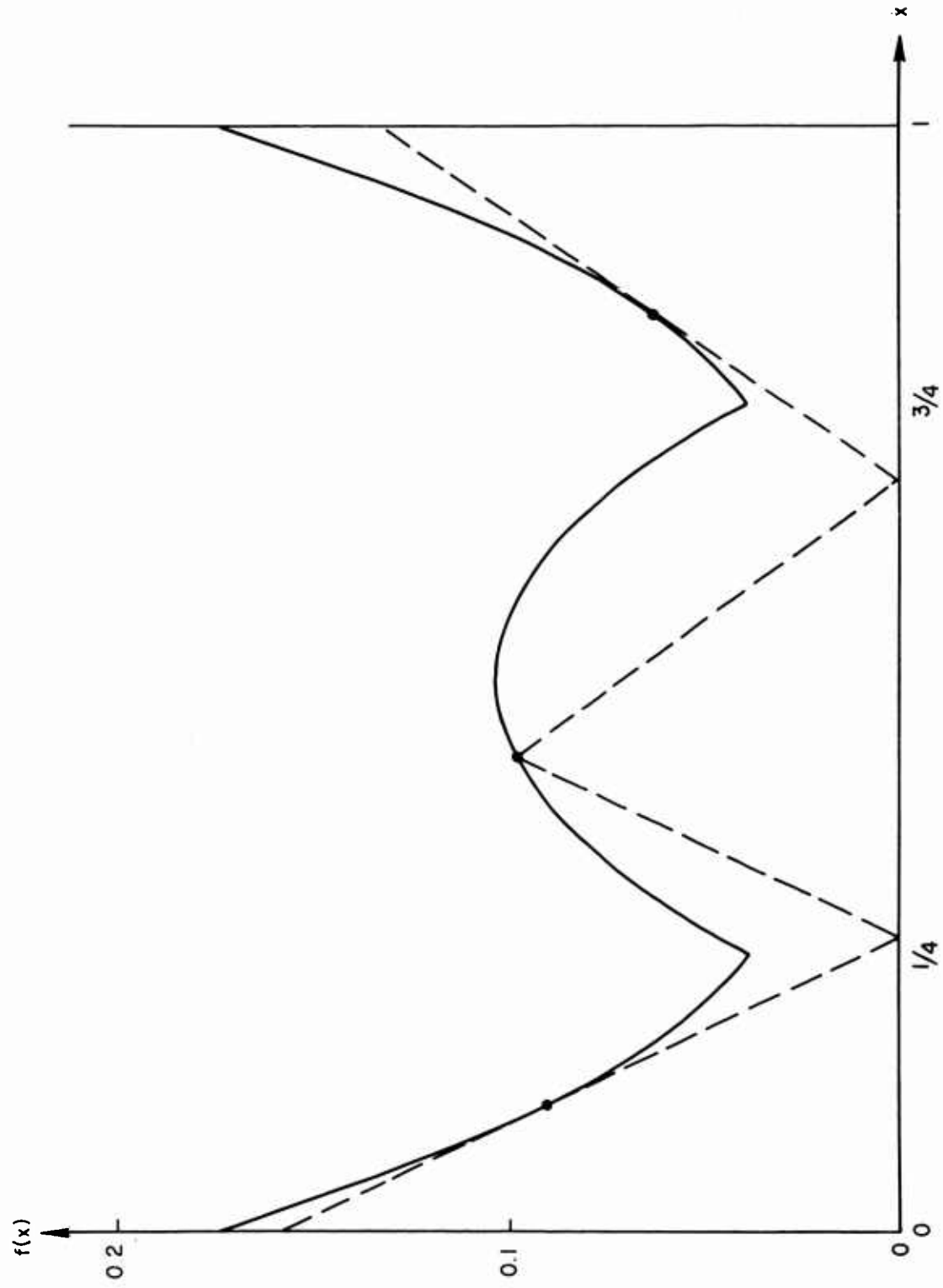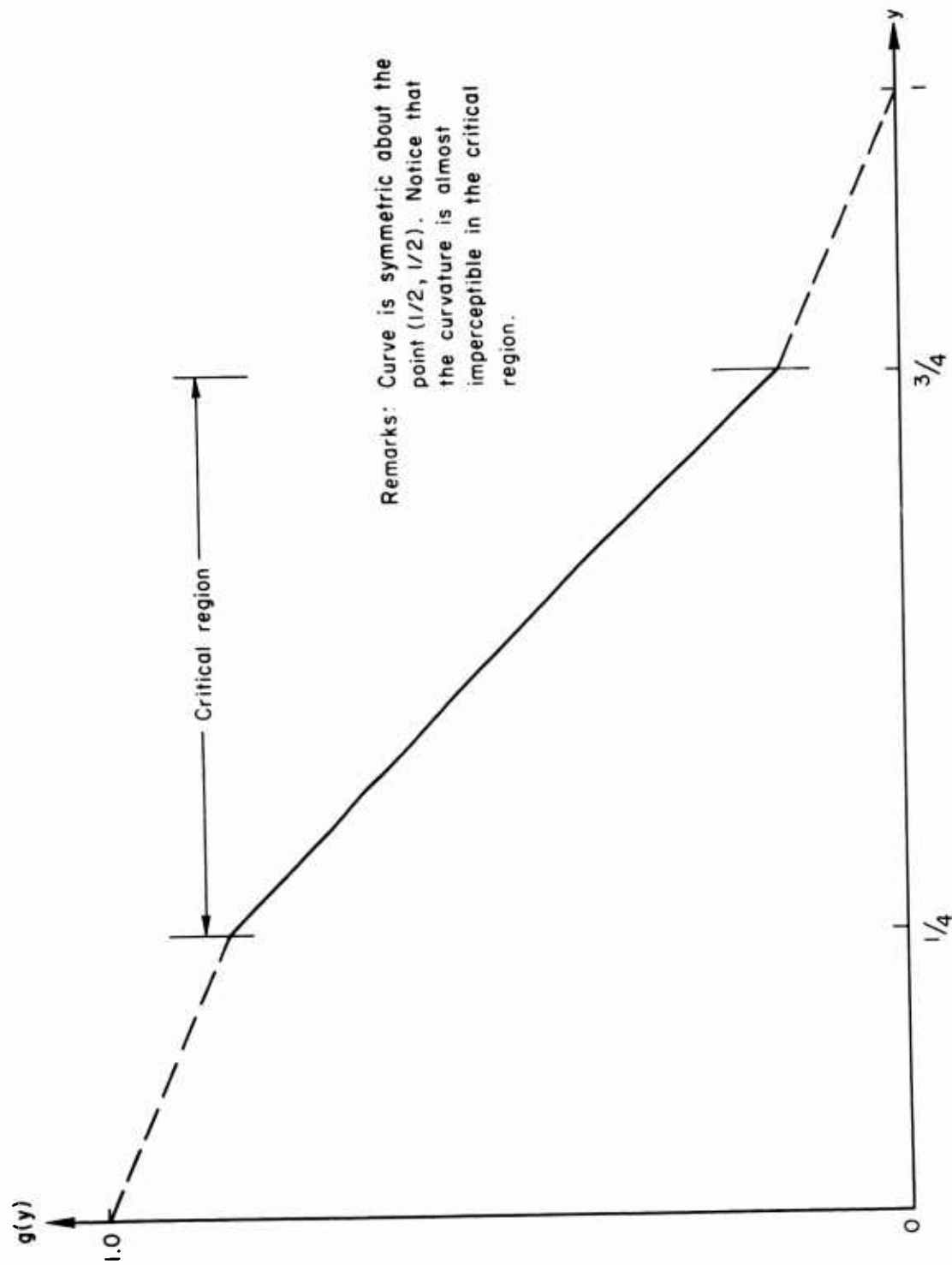
Fig. 2—Design moment of beam

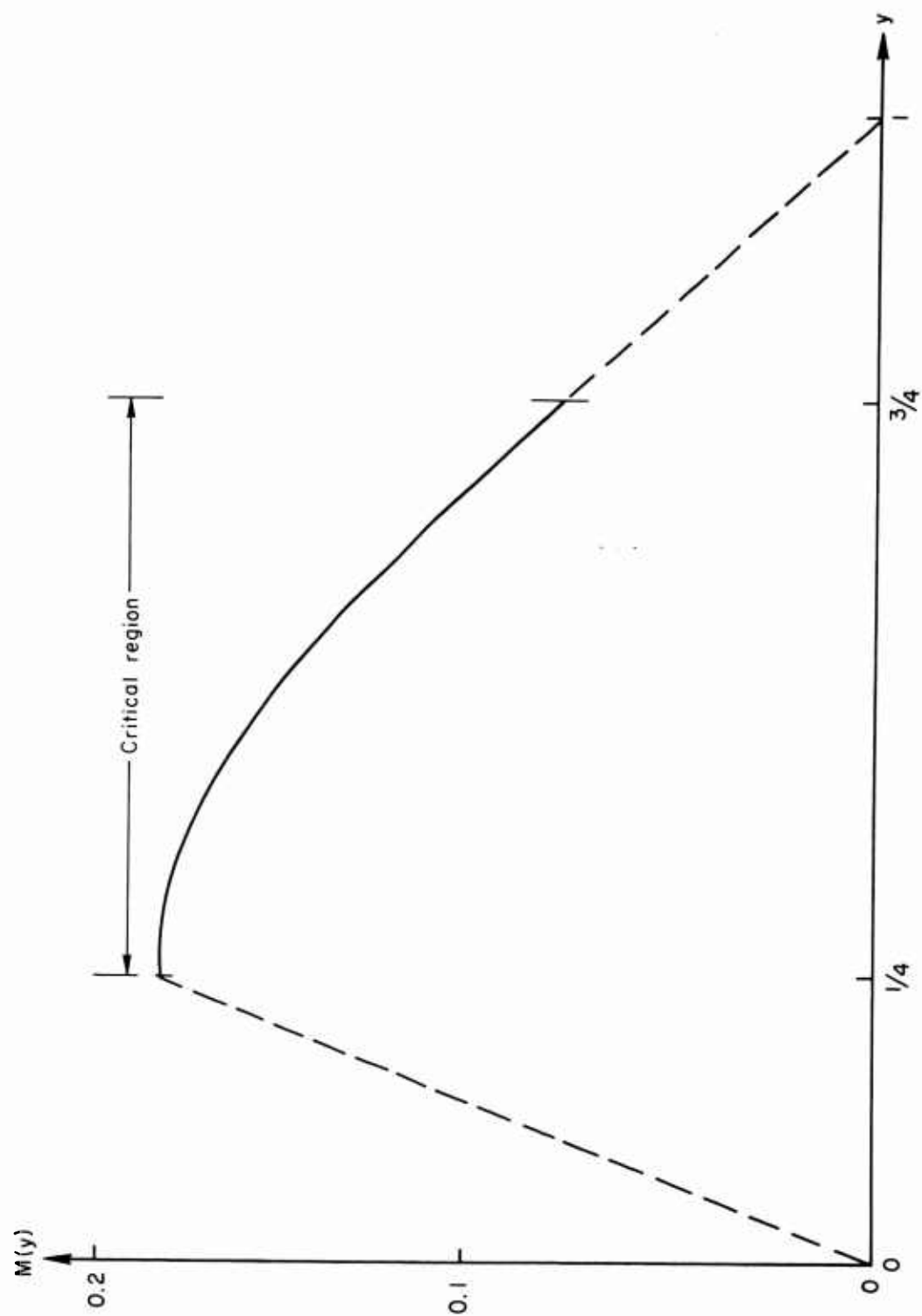Fig. 3—Vertical reaction at left end of beam

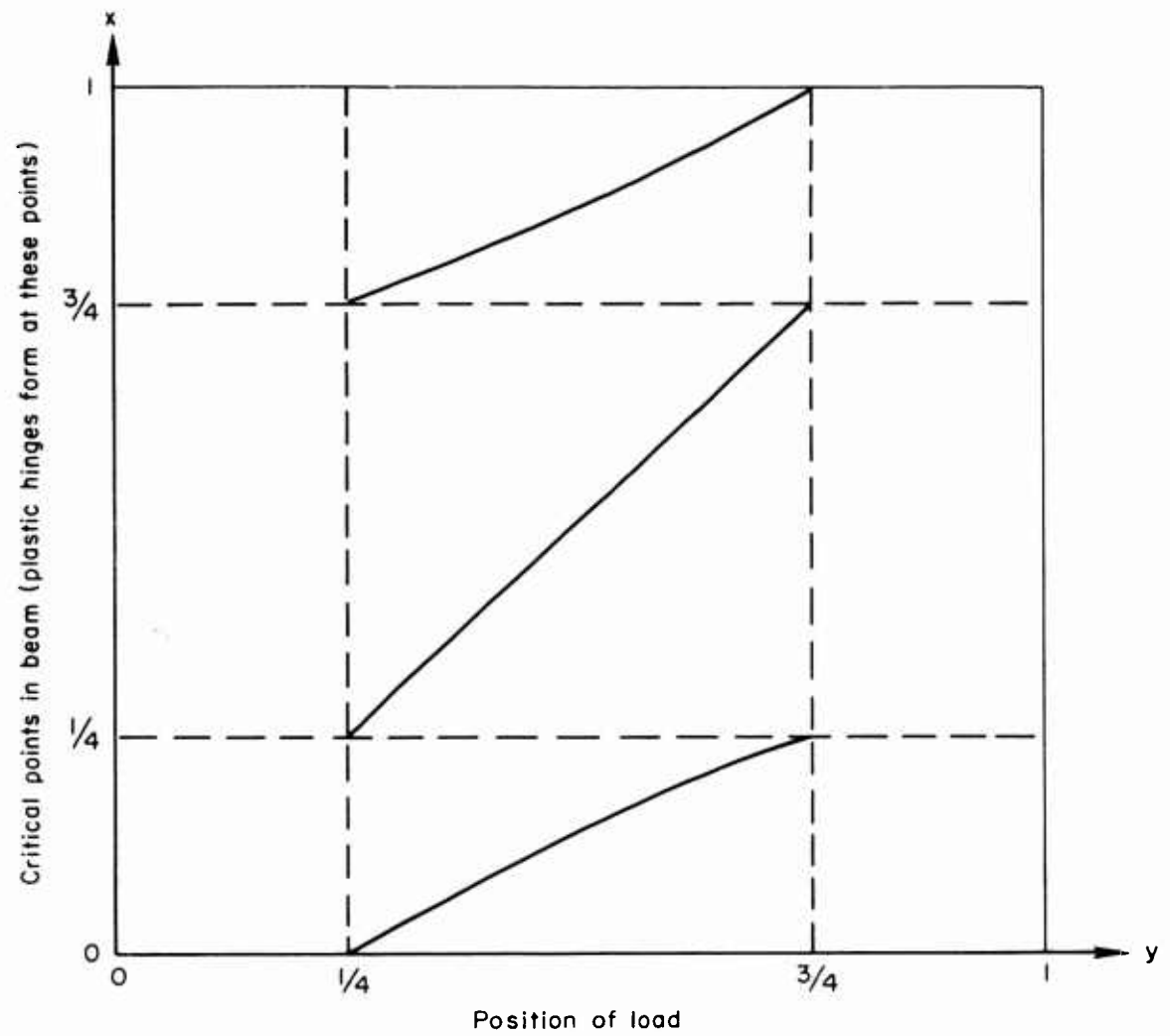Fig. 4—Moment reaction at left end of beam

Fig. 5—Critical points in beam versus load position

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1. Gross, O., and W. Prager, <u>Minimum—Weight Design for Moving Loads</u>, The RAND Corporation, Research Memorandum RM—2887—PR, January, 1962.

2. Gross, O., <u>A Linear Program of Prager's: Notes on Linear Programming and Extensions</u> — Part 60, The RAND Corporation, Research Memorandum RM—2993—PR, April, 1962.